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SELF-SIMILAR PROPAGATION REGIMES OF A NONSTATIONARY HIGH-TEMPERATURE CONVECTIVE JET IN THE ADIABATIC ATMOSPHERE

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An integrated model of a nonstationary high-temperature convective jet that includes the universal dependence of the upper boundary of the convective front on the power of a point heat source is proposed. A class of self-similar solutions corresponding to heat sources whose power changes in time according to the power and exponential laws is considered. Calculation results are compared with known experimental vertical-velocity and temperature profiles on the jet axis.

Introduction. A theoretical investigation of nonstationary convective jets has been initiated fairly recently [1]. In [1], the use of the approximation of a vertical boundary layer and the Kármán–Pohlhausen method allowed one to write amplitude equations for the vertical velocity and temperature on the jet axis. To close the system of equations, the heuristic evolutionary differential equation for the jet radius was used. A similar approach was applied in [2]. Within the framework of these models, a class of self-similar solutions corresponding to a point heat source whose power changes in time according to a power law was constructed in [1, 2]. Calculation results were compared with experimental data.

Another hydrodynamic description of the nonstationary convective jet is given in [3]. The Prandtl approach is used as a basis of the model. To determinate the plane (upper) boundary of the jet, which corresponds to the base of a cone, the universal equation of convective-front propagation [3, 4], which relates the jet height to the time dependence of the power of a point heat source, was used. In [3, 4], it was shown that in the given model, there are also self-similar regimes corresponding to heat sources whose power changes in time according to a power law. It is important that the universal equation of convective-front propagation allows one to construct a self-similar solution corresponding to an exponential heat source, which can be considered an envelope of the family of power solutions. The constructed solution is a self-similar solution of the second kind [5], because it cannot be obtained within the framework of the theory of dimensions.

In the present work, the integrated hydrodynamic model [3] is refined owing to a high-temperature generalization and the use of experimental horizontal temperature and vertical-velocity profiles [6].

1. The Problem of a Turbulent Jet above a Point Heat Source. We consider the problem of propagation of an axisymmetrical convective jet in the adiabatic atmosphere above a point heat source. Let t be the time, $(r,\, \varphi,\, z)$ be the cylindrical coordinate system whose z axis is directed opposite to the acceleration of gravity g.

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To describe the jet propagation, the "deep"-convection equations of an ideal gas [7], which generalize the Boussinesq model owing to the partial allowance for air compressibility, are used. Let $\Theta_{\rm a} = \text{const}$ be the static value of the potential temperature of dry air in the quiet atmosphere, and Θ be the local potential air temperature.[3]

Following [7], we introduce the local dimensionless potential temperature $\theta = (\Theta - \Theta_{\rm a})/\Theta_{\rm a}$.

The dimensionless density of dry quiet air in the adiabatic atmosphere $\bar\rho_{\rm a}$ is set by the relations

$$\frac{1}{\bar\rho_{\rm a}}\,\frac{d\bar\rho_{\rm a}}{dz} = -\frac{c_p - R_d}{R_d}\,\frac{g}{c_p}\,\frac{1}{T_{\rm a}}, \qquad \bar\rho_{\rm a}(0) = 1, \qquad \frac{dT_{\rm a}}{dz} = -\frac{g}{c_p},$$

where R_d and c_p are, respectively, the gas constant and the thermal capacity of dry air at constant pressure; $T_{\rm a}$ is the air temperature in the adiabatic atmosphere.

The flow in an axisymmetrical nonstationary convective jet is considered in the approximation of a vertical boundary layer [8]:

$$\begin{gathered}\frac{\partial}{\partial t}\,\bar\rho_{\rm a} w + \frac{1}{r}\,\frac{\partial}{\partial r}\,\bar\rho_{\rm a} uwr + \frac{\partial}{\partial z}\,\bar\rho_{\rm a} ww = g\bar\rho_{\rm a}\theta + \frac{1}{r}\,\frac{\partial}{\partial r}\Big(\nu_w\bar\rho_{\rm a} r\,\frac{\partial w}{\partial r}\Big),\\ \frac{\partial}{\partial t}\,\bar\rho_{\rm a}\theta + \frac{1}{r}\,\frac{\partial}{\partial r}\,\bar\rho_{\rm a} u\theta r + \frac{\partial}{\partial z}\,\bar\rho_{\rm a} w\theta = \frac{1}{r}\,\frac{\partial}{\partial r}\Big(\nu_\theta\bar\rho_{\rm a} r\,\frac{\partial\theta}{\partial r}\Big), \quad \frac{1}{r}\,\frac{\partial}{\partial r}\,\bar\rho_{\rm a} ur + \frac{\partial}{\partial z}\,\bar\rho_{\rm a} w = 0.\end{gathered} \tag{1.1}$$

Here u and w is the velocity component along the r and z axes, respectively, and ν_w and ν_θ are the turbulent-exchange coefficients of the vertical velocity and the dimensionless potential temperature, respectively.

System (1.1) is considered in the domain $V = \{0 \leqslant r < \infty,\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant z \leqslant \infty\}$, where the infinite upper boundary is at a distance equal to the height of the adiabatic atmosphere. For $\bar\rho_{\rm a} = 1$, the "deep"-convection equations (1.1) become the Boussinesq equations.

The presence of the total kinetic and potential energies and the entropy in the set of "deep"-convection equations [7], i.e., the existence of integrals of the form

$$2\pi\int\limits_0^\infty \bar\rho_{\rm a}\Big(\frac{1}{2}\,w^2 - g\theta z\Big) r\,dr, \qquad 2\pi\int\limits_0^\infty \bar\rho_{\rm a}\theta r\,dr,$$

allows one to consider that the functions $\bar\rho_{\rm a} w^2$, $\bar\rho_{\rm a} g\theta z$, and $\bar\rho_{\rm a}\theta$ belong to the functional space $L_1(V)$ (see [9]). The integrability of the functions over the unlimited domain V results in the condition of their damping at indefinitely remote boundaries.

Taking into account that the medium is not disturbed, we set the no-flow and flow-attenuation conditions at the upper boundary of the domains in the form

$$\begin{aligned}&\lim_{z\to\infty} w(r,z,t) = 0, &&\lim_{z\to\infty} 2\pi\int\limits_0^\infty w(r,z,t) r\,dr = 0,\\ &\lim_{z\to\infty} ww(r,z,t) = 0, &&\lim_{z\to\infty} 2\pi\int\limits_0^\infty ww(r,z,t) r\,dr = 0,\\ &\lim_{z\to\infty} w\theta(r,z,t) = 0, &&\lim_{z\to\infty} 2\pi\int\limits_0^\infty w\theta(r,z,t) r\,dr = 0.\end{aligned} \tag{1.2}$$

At the lateral surface, we adopt the following no-flow and flow-attenuation conditions:

$$\lim_{r\to\infty} \bar\rho_{\rm a} ur = 0, \quad \lim_{r\to\infty} \nu_w\bar\rho_{\rm a} r\,\frac{\partial w}{\partial r} = 0, \quad \lim_{r\to\infty} \nu_\theta\bar\rho_{\rm a} r\,\frac{\partial\theta}{\partial r} = 0. \tag{1.3}$$

[3]For an ideal gas, the potential temperature Θ is determined by the relation $\Theta = T(p/p_{\rm n})^{-R_d/c_p}$, where T is the local gas temperature, p is the local pressure, $p_{\rm n}$ is the constant normal gas pressure at the underlying surface, which is approximately equal to 1 atm, and R_d and c_p are, respectively, the gas constant and the thermal capacity at constant pressure.

At the lower boundary of the domain, we set a point nonstationary heat source and a zero shock source, i.e.,

$$\lim_{z\to 0}[\bar\rho_{\rm a} w\theta(r,z,t)] = \frac{1}{2\pi r}\,S_0(t)\delta(r), \quad \lim_{z\to 0}[\bar\rho_{\rm a} ww(r,z,t)] = \frac{1}{2\pi r}\,P_0(t)\delta(r). \tag{1.4}$$

Here $P_0(t) = 0$ and $S_0(t) > 0$ are the powers of the point shock and heat source and $\delta(r)$ is the Dirac delta-function.

As the initial condition for $t = t_0$, we adopt the undisturbed-atmosphere state

$$w(r,z,t_0) = 0, \qquad \theta(r,z,t_0) = 0. \tag{1.5}$$

Relations (1.1)–(1.5) form a closed set of equations.

2. The Integrated Model of a Convective Jet above a Point Heat Source. To construct an approximate solution of system (1.1), the Kármán–Pohlhausen integrated method [8] is used. It is assumed that the unknown functions in the field of ascending motion $0 < r < R(z,t)$ can be approximated by relations with separable variables:

$$w(r,z,t) = \tilde w(z,t) f_w\Big(\sqrt{\bar\rho_{\rm a}}\,\frac{r}{R}\Big), \quad u(r,z,t) = -\frac{1}{\bar\rho_{\rm a}}\,\frac{\partial \bar\rho_{\rm a}\tilde w(z,t)}{\partial z}\,\frac{1}{r}\int_0^r r f_w\Big(\sqrt{\bar\rho_{\rm a}}\,\frac{r}{R}\Big)dr,$$
$$\theta(r,z,t) = \tilde\theta(z,t) f_\theta\Big(\sqrt{\bar\rho_{\rm a}}\,\frac{r}{R}\Big). \tag{2.1}$$

Here $\tilde w(z,t)$ and $\tilde\theta(z,t)$ are, respectively, the vertical velocity and the dimensionless potential temperature on the jet axis and f_w and f_θ are specified functions.

For comparison of the given and existing models [1, 2], we use the exponential approximations of the parameter profiles in accordance with known experimental data [6]:

$$f_w(\xi) = \exp(-\lambda_w\xi^2), \quad f_\theta(\xi) = \exp(-\lambda_\theta\xi^2), \quad \xi = \sqrt{\bar\rho_{\rm a}}\,r/R. \tag{2.2}$$

Substituting (2.1) and (2.2) into Eqs. (1.1) and integrating the resulting equations over the transverse cross-sectional area of the jet, we obtain

$$\frac{\partial}{\partial t}\,\tilde w R^2 + \frac{1}{2}\,\frac{\partial}{\partial z}\,\tilde w\tilde w R^2 = \alpha_g g\tilde\theta R^2, \quad \frac{\partial}{\partial t}\,\tilde\theta R^2 + \frac{1}{1+\alpha_g}\,\frac{\partial}{\partial z}\,\tilde w\tilde\theta R^2 = 0, \tag{2.3}$$

where $\alpha_g = \lambda_w/\lambda_\theta$ is a constant factor. Equations (2.3) should be supplemented by the boundary conditions

$$\lim_{z\to 0}[\tilde w\tilde w R^2(z,t)] = 0, \qquad \lim_{z\to 0}[\tilde w\tilde\theta R^2(z,t)] = \frac{1}{\pi}\,\frac{\lambda_w}{k^2}\,S_0(t), \qquad k^2 = \frac{\alpha_g}{1+\alpha_g}. \tag{2.4}$$

To close system (2.3), (2.4), it is necessary to use the equation for the jet radius R. Following [3], we assume that the convective thermic is approximated by the conic surface and the head part, the form of the thermic remaining constant at any moment of time (Fig. 1).

According to the Prandtl hypothesis, the following law of linear extension of the jet above a point source, which are adopted in the models from [3, 10], is used:

$$R(z,t) = \alpha_R z, \qquad 0 \leqslant z \leqslant h(t). \tag{2.5}$$

Here α_R is the angle jet-expansion coefficient whose magnitude varies from 0.1 to 0.2 (see, e.g., [10]) and $h(t)$ is the height of the upper boundary of the conic surface of the jet. Within the framework of the adopted formulation of the problem, the motion in the region $z > h(t)$ is not considered.

As an equation that describes the propagation of the upper boundary of a convective jet from a heat source in the neutral atmosphere, we use the relation [3, 4]

$$\lambda_0^2 h^2\Big(\frac{dh}{dt}\Big)^2 = g\int_{t_0}^{t} S_0(t)\,dt, \qquad \lambda_0^2 = \frac{\pi}{2\lambda_w}\,\alpha_R^2(1+\alpha_g)^2. \tag{2.6}$$

In subsequent numerical experiments, it will be assumed that, according to [6], we have $\lambda_w = 96\alpha_R^2$, $\lambda_\theta = 0/74\lambda_w$ and, hence, $\alpha_g = \lambda_w/\lambda_\theta = 1.35$. Here $\lambda_0^2 = 9.04\cdot 10^{-2}$, which corresponds, in order of magnitude, to the experimental values of $\lambda_0^2 = (2.22–4.56)\cdot 10^{-2}$ from [11].

Relations (2.3)–(2.6) form a closed system of the integrated model of a vertical convective jet.

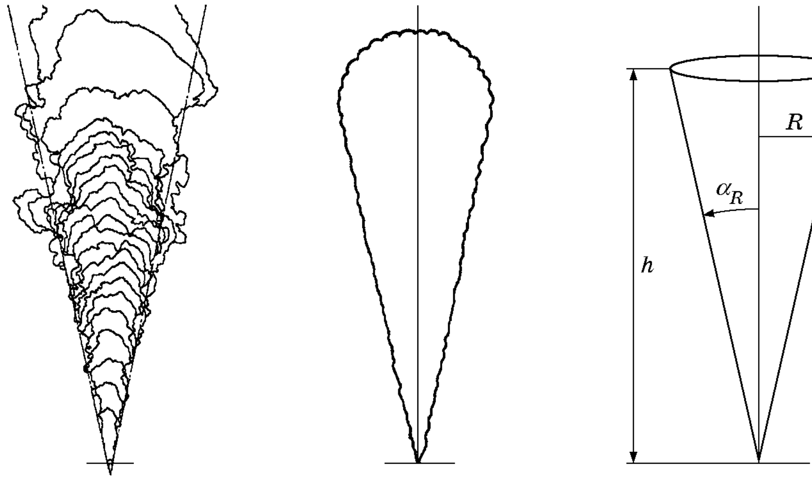

Fig. 1. Contour of a developing convective jet corresponding to a point heat source and its approximation.

3. Quasistationary Equations as the Asymptotic Behavior of the Solution near a Source. Taking into account the presence of the singularity of the solution at the coordinate origin, which is connected with the effect of the point heat source, it is expedient to describe the asymptotic solution of the problem of a convective jet near a source. According to [1, 2], the asymptotic behavior of the solution is described by the quasistationary equations (2.3), in which there are no time derivatives.

Let the power of the reduced heat source $\tilde{S}$ be given by the relation $\tilde{S}(t) = \lambda_w S_0(t)/(\pi k^2)$. The corresponding parameters of the quasistationary jet have the form

$$\tilde{w}_{\rm a}(z,t) = ((3/2)\alpha_g g\tilde{S}(t))^{1/3}\alpha_R^{-2/3}z^{-1/3},$$
$$\tilde{\theta}_{\rm a}(z,t) = ((3/2)\alpha_g g\tilde{S}(t))^{-1/3}\tilde{S}(t)\alpha_R^{-4/3}z^{-5/3}, \qquad R(z,t) = \alpha_R z, \tag{3.1}$$

where $\tilde{w}_{\rm a}$ and $\tilde{\theta}_{\rm a}$ are, respectively, the vertical velocity and the dimensionless potential temperature on the axis of the quasistationary jet in the adiabatic atmosphere.

It is noteworthy that relations (3.1) are of independent interest, because in the case $\tilde{S}(t) = \text{const}$, they correspond to the point solution of the problem of stationary-jet propagation in the neutral atmosphere. The functional dependences (3.1) were obtained for the first time by Zel'dovich [12] within the framework of the similarity theory. In [10], relations of the type (3.1) are found as solutions of the integrated model of a jet.

Let $h(t)$ be the height of a convective jet which corresponds to a heat source of power $S_0(t)$ and which is calculated according to (2.6). We introduce the dimensionless variable $\eta = z/h(t)$. Then, the quasistationary solution (3.1) can be presented in the form

$$\tilde{w}_{\rm a}(z,t) = \frac{dh}{dt}\,w_*^s(\eta,t), \quad \tilde{\theta}_{\rm a}(z,t) = \frac{1}{gh}\Big(\frac{dh}{dt}\Big)^2\theta_*^s(\eta,t), \quad R(z,t) = hr_*(\eta,t). \tag{3.2}$$

Here the dimensionless functions w_*^s, θ_*^s, and r_* and the normalized power of the heat flux S_* have the forms

$$w_*^s(\eta,t) = \frac{1}{\alpha_R\eta}\Big(\frac{3}{2}\,\alpha_g\alpha_R S_*\eta^2\Big)^{1/3}, \qquad \theta_*^s(\eta,t) = \frac{S_*}{\alpha_R\eta}\Big(\frac{3}{2}\,\alpha_g\alpha_R S_*\eta^2\Big)^{-1/3},$$
$$r_*(\eta,t) = \alpha_R\eta, \qquad S_*(t) = \frac{1}{\pi}\,\frac{\lambda_w}{k^2}\,gS_0(t)\Big(h\Big(\frac{dh}{dt}\Big)^3\Big)^{-1}, \tag{3.3}$$

respectively.

For comparison with experimental data, the general solution of the nonstationary problem is referred to the quasistationary solution (3.1)–(3.3).

4. Self-Similar Regimes of Convective-Front Propagation. For a number of specially specified amplitudes of the heat source, the corresponding regimes of convective-front propagation can be obtained within the framework of the theory of dimensions without using the universal relation (2.6). We show that Eq. (2.6) not only includes all the known self-similar dependences as particular solutions, but also allows one to construct new self-similar regimes.

We consider the convection caused by a power heat source: $t_0 = 0$, $S_0(t) = Q_q q t^{q-1}$, $q > 0$, and $Q_q = \text{const}$. From (2.6) follows the self-similar dependence

$$h(t) = \left(\left(\frac{2}{\lambda_0}\right)^2 \frac{gQ_q}{(q/2+1)^2}\, t^{q+2}\right)^{1/4}. \tag{4.1}$$

Let us consider the convection caused by an exponential heat source: $t_0 = -\infty$ and $S_0(t) = Q_\infty q \exp(qt)$ ($Q_\infty = \text{const}$). From (2.6) follows

$$h(t) = \left(\left(\frac{2}{\lambda_0}\right)^2 \frac{gQ_\infty}{(q/2)^2} \exp(qt)\right)^{1/4}. \tag{4.2}$$

Relation (4.2) is self-similar, although it cannot be obtained from the theory of dimensions. It should be interpreted as an envelope of the family of power solutions (4.1) as $t_0 \to -\infty$ and $q \to +\infty$.

5. Development of Self-Similar Jets above a Point Heat Source. The integrated model of a nonstationary jet and corresponding self-similar solutions for point heat sources whose power changes in time under a power law were obtained for the first time in [1, 2]. We show that in the case of point heat sources, the self-similar regimes (4.1) and (4.2) also generate corresponding classes of self-similar solutions for the integrated jet model proposed.

Let $\eta = z/h(t)$ is a dimensionless parameter. For $0 < z < h(t)$, the self-similar solution of system (2.3)–(2.6) can be searched for in the form

$$\tilde{w}(z,t) = \frac{dh}{dt}\, w_*(\eta), \quad \tilde{\theta}(z,t) = \frac{1}{gh}\Big(\frac{dh}{dt}\Big)^2 \theta_*(\eta), \quad R = hr_* = h\alpha_R\eta, \tag{5.1}$$

where w_*, θ_*, and r_* are dimensionless functions.

For power sources (4.1), substitution of (5.1) in system (2.3) for $0 < \eta < 1$ leads to a system of ordinary differential equations

$$\Big(2 + \frac{q-2}{q+2}\Big) w_* r_*^2 - \eta\,\frac{d}{d\eta}\,(w_* r_*^2) + \frac{1}{2}\,\frac{d}{d\eta}\, w_* w_* r_*^2 = \alpha_g \theta_* r_*^2, \tag{5.2}$$

$$\Big(1 + 2\,\frac{q-2}{q+2}\Big) \theta_* r_*^2 - \eta\,\frac{d}{d\eta}\,(\theta_* r_*^2) + \frac{1}{1+\alpha_g}\,\frac{d}{d\eta}\, w_* \theta_* r_*^2 = 0, \quad r_* = \alpha_R \eta.$$

According to (2.4), the boundary conditions of system (5.1) take the form

$$\lim_{\eta\to 0} (w_* w_* r_*^2) = 0, \quad \lim_{\eta\to 0} (w_* \theta_* r_*^2) = \frac{1}{\pi}\,\frac{\lambda_w}{k^2}\,\frac{4q}{q+2}\,\lambda_0^2. \tag{5.3}$$

Similar relations can be written for the exponential source (4.2) as well. Here the coefficients in the corresponding equations (5.2) and (5.3) are obtained by means of the limit transition as $q \to \infty$. In the case of the exponential source (4.2), the solution is a self-similar solution of the second kind [5], because it cannot be obtained on the basis of the theory of dimensions.

We note that in the domain $10 < q < \infty$, the coefficients (5.2) and (5.3) can be considered almost constant. Therefore, all the self-similar jets with quite large values of q have almost identical velocity and temperatures profiles corresponding to the exponential source.

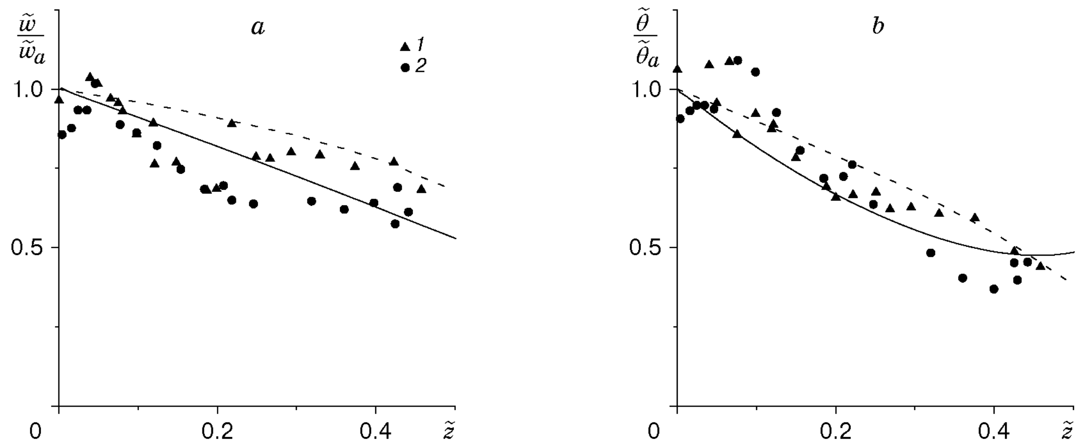

Fig. 2. Distributions of the normalized vertical velocities (a) and the potential temperature (b) along the $\tilde{z}$ axis: solid curves refer to calculations by means of the proposed model and dashed curves refer to calculations according to the model [2]; curves 1 and 2 refer to the data for two series of measurements in [2], in which the heat sources differ by the method of packing the combustibles.

6. Numerical Description of Self-Similar Regimes of Jet Development. Taking into account the asymptotic behavior of the convective jet for $\eta \ll 1$, according to [1, 2], for representation of the results, we use the functions

$$\tilde{w}/\tilde{w}_{\rm a} = w_*/w_*^s = \varphi_w(\eta), \quad \lim_{\eta\to 0} \varphi_w(\eta) = 1, \qquad \tilde{\theta}/\tilde{\theta}_{\rm a} = \theta_*/\theta_*^s = \varphi_\theta(\eta), \quad \lim_{\eta\to 0} \varphi_\theta(\eta) = 1.$$

According to [1, 2], we introduce the dimensionless parameter

$$\tilde{z} = \frac{1}{S_0(t)} \frac{dS_0}{dt} \frac{z}{\tilde{w}_{\rm a}(z,t)}.$$

Using relation (3.1), one can show that, for the power heat sources (4.1), we have

$$\tilde{z} = C\eta^{4/3}, \qquad C = \frac{4}{1+\alpha_g} \frac{q-1}{q+2}\Big(\frac{q+2}{3q}\Big)^{1/3}.$$

As an example, the numerical solution of the self-similar equations for $q = 4$ and $\alpha_R = 0.1$ is considered. Calculation results obtained for the normalized vertical velocities and the potential temperature are compared with experimental data and calculation results obtained by means of the model from [2] in Fig. 2.

Conclusions. The results of the present study show that the proposed integrated model of a convective jet that includes the universal equation of propagation of the upper boundary of a convective front contains a class of self-similar solutions corresponding to power and exponential heat sources. The calculation results agree well with known experimental data.

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